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1993 J. Phys. A: Math. Gen. 26 665

(http://iopscience.iop.org/0305-4470/26/3/024)

J. Phys. A: Math. Gen. 26 (1993) 665–679. Printed in the UK

Symmetries and integrability of generalized diffusion reaction equations

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Received 16 March 1992, in final form 28 September 1992

Abstract. Using Lie group methods and the Painlevé test, we analyse nonlinear diffusion reaction equations $u_t = \nabla(D(u)\nabla u) + f(u)$ with power law diffusion coefficients ($D \sim u^\nu$) and arbitrary nonlinear reaction terms $f(u)$ which have a wide spectrum of applications in many areas of science. The Lie-group-based similarity method leads to a classification of the reaction terms according to its symmetry properties. With the help of the adjoint representation, the optimal system of similarity reductions is calculated. To check the integrability of the partial differential equation, the existence of generalized (Lie–Bäcklund) symmetries is investigated. Apart from three known cases, no further cases with third-order symmetries exist. Examining the integrability of the second-order ordinary differential equations resulting from the reductions, only a few parameter combinations can be found for which the Painlevé property is given. However, we are able to construct unknown integrals of motion for a much larger range of parameter values. From the integrals, exact similarity solutions may be derived. This is demonstrated by examples corresponding to the important moving-wave reduction.

1. Introduction

An important class of nonlinear evolution equations are the so-called diffusion reaction equations

$$u_t = \mathrm{div}[D(u)\,\mathrm{grad}\, u] + f(u)\,. \tag{1}$$

In (1), the first term on the right-hand side describes diffusion with a (generally non-constant) diffusion coefficient $D(u)$, whereas the second term, the reaction term, is related to source and loss processes. Equation (1) has a wide range of applications in physical and related sciences, e.g. in biophysics [1,2], plasma physics [3,4], solid state physics [5], hydrodynamics [6], and chemical reactor design [7].

In this paper, we will discuss (1) where the diffusion coefficient has a power law dependency

$$D(u) = u^\nu \tag{2}$$

with real exponent ν. The case $\nu > 0$ corresponds to 'slow diffusion', whereas $\nu < 0$ leads to 'fast diffusion'. The reaction term $f(u)$ in (1) is considered to be a general

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function of the concentration u. Using the dimensionality parameter λ ($\lambda = 1, 2, 3$) and assuming cylindrical or spherical symmetry for the two- and three-dimensional cases, (1) reads

$$u_t = r^{1-\lambda}\left[r^{\lambda-1}\,u^{\nu}\,u_r\right]_r + f(u)\,. \tag{3}$$

Equations of the form (3) with $f \equiv 0$, so-called nonlinear diffusion equations, were recently investigated with respect to Painlevé property [8], first integrals [9], and exact similarity solutions [8, 10–12]. In order to study effects resulting from combinations of nonlinear diffusion and reaction terms, we will consider $\nu \neq 0$ and $f(u) \not\equiv 0$ in (3).

Section 2 is concerned with the symmetry analysis of (3). The classical similarity method [13, 14] leads to a classification of the reaction terms $f(u)$. If $f(u)$ is a power of u or a combination of a linear term and a term proportional to $u^{\nu+1}$, a large variety of symmetry transformations is found whereas other forms of $f(u)$ only allow translational symmetries. The results presented here are generalizations for two- and three-dimensional radial symmetric problems (i.e. $\lambda = 2, 3$) of those given by Galaktionov *et al* [15]. In [15], a symmetry analysis is performed up to the point of determining the infinitesimal representation of the symmetry groups in the one-dimensional case ($\lambda = 1$). In the present work, we calculate the symmetry groups for all possible values of the dimension λ including the corresponding finite symmetry transformations which allow the construction of new solutions from known ones. Furthermore, by applying the adjoint representation, the appropriate optimal system of similarity reductions is determined. The most important differential equations resulting from the reductions can be summarized in a general but single equation. A useful criterion for integrability of partial differential equations having no Hamiltonian structure is the existence of infinitely many Lie–Bäcklund symmetries. For (3) at least a third-order Lie–Bäcklund symmetry must exist. Looking for such generalized symmetries, three cases are found which are already known in the literature [6, 15].

In the remaining part of the paper, the general second-order ordinary differential equation resulting from the reductions is investigated in detail. Section 3 deals at first with integrability with respect to the Painlevé property. At the beginning of this century, Painlevé *et al* (cf [16]) showed that each second-order differential equation with the Painlevé property can be transformed into one of 50 standard forms. The general solutions are, according to the definition of the Painlevé property, meromorphic functions of the two integration constants. We determine parameter values for which transformation to a standard form is possible and, therefore, an explicit general solution is known. It turns out that the Painlevé property is satisfied only for a small set of parameter values. With the help of the transformation and the solution of the standard equation, the general solution of the Painlevé cases can be obtained. Apart from this Painlevé analysis for ordinary differential equations, additionally a Painlevé test based on the WTC algorithm [17] is carried out for partial differential equations of type (3). For the equations under consideration the test reveals a negative result.

Not only for the Painlevé cases, but also for a much larger class of the second-order differential equations, a first integration can be performed. By multiplying the equation by a suitable expression and by collecting terms forming a total differential, first integrals are constructed. For special values of the integration constant, the second integration can be carried out leading to explicit particular solutions with one free constant. The second integration for arbitrary values of the integration constant, however, is only possible in a more restricted parameter space. Finally it is demonstrated how explicit solutions of (3) are obtained from the first integrals and the standard forms. Two examples, both applying to the important reduction with the moving-wave variable, are considered. If $f(u) = m =$ constant and $\nu = -1$, a pulse-shaped moving-wave solution of (3) is given, depending on three arbitrary parameters. The second example deals with a generalization of Fisher's equation. The solution is a kink-shaped wavefront with an edge at its front part.

2. Symmetries and similarity reductions

Classical similarity analysis [13, 14] determines transformations which leave the differential equation invariant. In the infinitesimal representation, the corresponding generator of the transformation is written by

$$X = \tau(t,r,u)\frac{\partial}{\partial t} + \xi(t,r,u)\frac{\partial}{\partial r} + \eta(t,r,u)\frac{\partial}{\partial u}. \tag{4}$$

The infinitesimals τ, ξ, and η of a differential equation $\Delta = 0$ of the order n are calculated from the condition of invariance

$$X^{(n)}\,\Delta|_{\Delta=0} = 0 \tag{5}$$

in an algorithmic way. In (5), $X^{(n)}$ is the nth extension of the generator X. The infinitesimals τ, ξ of (3) with non-constant diffusion coefficients ($\nu \neq 0$) are listed in table 1 where d, m, g, σ are constants characterizing the reaction term and b, c, k, h, s, q are constants of the symmetry group. The infinitesimal η is given by

$$\eta = (2\xi_r - \tau_t)u/\nu\,. \tag{6}$$

The values $\nu = -\frac{4}{3}$, $\lambda = 1$ and $\nu = -1$, $\lambda = 2$ play a special role. Their appearance is connected with the power law diffusion coefficient $D(u) = u^\nu$ and they are also found in the case of vanishing reaction term $f(u) \equiv 0$. With respect to the reaction term, three different types need to be distinguished. Hence, the similarity method leads to a classification of the reaction term according to its symmetry properties. The three classes of $f(u)$ are given by:

(i) $f(u) = -(d/\nu)\,u + m\,u^{\nu+1}$: these reaction terms allow a large variety of transformations. Depending on the dimension λ and the values of ν, d, m (describing diffusion and reaction behaviour), symmetry groups with two to five group constants emerge;

(ii) $f(u) = gu^{\sigma+1}$: here, the symmetry groups depend on 2 or 3 group constants. Especially for this reaction term, a scaling symmetry is possible;

(iii) other $f(u)$: only a small number of allowed transformations exist, leading to symmetry groups with 1 or 2 group constants.

The generators of the symmetry groups of (3) with the corresponding global transformations are shown in table 2. The transformations allow the determination of new solutions $u_1(r_1, t_1)$ from known solutions $u(r,t)$. Taking the conditions for the appearance of the corresponding generators X_i (cf table 2, into account, the variables are connected by

$$(u_1, r_1, t_1) = e^{\epsilon X_i}(u, r, t)\,. \tag{7}$$

Table 1. Infinitesimals for $u_t = r^{1-\lambda}\left[r^{\lambda-1}u^{\nu}u_r\right]_r + f(u)$.

λ	ν	$f(u)$	$\tau(t)$	$\xi(r)$
1	$-\frac{4}{3}$	$\frac{3d}{4}u$	$c-\frac{k}{d}e^{dt}$	$h\,r^2+s\,r+q$
		$m\,u^{-1/3}$	$c-bt$	$q+\frac{a_1}{\omega}\cos(\omega r)+\frac{a_2}{\omega}\sin(\omega r)$ [a]
		$\frac{3d}{4}u+m\,u^{-1/3}$	$c-\frac{k}{d}e^{dt}$	$q+\frac{a_1}{\omega}e^{\omega r}-\frac{a_2}{\omega}e^{-\omega r}$ [b]
	$\neq -\frac{4}{3}$	$-\frac{d}{\nu}u$	$c-\frac{k}{d}e^{dt}$	$sr+q$
		$m\,u^{\nu+1}$	$c-bt$	q
		$-\frac{d}{\nu}u+m\,u^{\nu+1}$	$c-\frac{k}{d}e^{dt}$	q
	Arbitrary	$g\,u^{\sigma+1}$	$c-bt$	$b\,\frac{\nu-\sigma}{2\sigma}\,r+q$
		Arbitrary	c	q
2	-1	du	$c-\frac{k}{d}e^{dt}$	$wr\ln r+sr$
2, 3	Arbitrary	$-\frac{d}{\nu}u$	$c-\frac{k}{d}e^{dt}$	sr
		$m\,u^{\nu+1}$	$c-bt$	0
		$-\frac{d}{\nu}u+m\,u^{\nu+1}$	$c-\frac{k}{d}e^{dt}$	0
		$g\,u^{\sigma+1}$	$c-bt$	$b\,\frac{\nu-\sigma}{2\sigma}\,r$
		Arbitrary	c	0

[a] $m < 0$ $(\omega = \sqrt{-4m/3})$
[b] $m > 0$ $(\omega = \sqrt{4m/3})$

The Lie algebras corresponding to the symmetry groups are characterized by the commutators of the generators for the allowed transformations. For reaction terms with $f(u) = -(d/\nu)\,u + m\,u^{\nu+1}$ and $f(u) = gu^{\sigma+1}$, the commutators are shown in tables 3 and 4, respectively. For any other $f(u)$, there are at most two commutating generators X_1 and X_2, i.e. $[X_1, X_2] = 0$. Here and in the following, the generators $X_6, \ldots, X_9$ corresponding to rather complicated transformations and appearing only for the special parameter combinations $\lambda = 1,\ \nu = -\frac{4}{3}$ or $\lambda = 2,\ \nu = -1$ (cf table 2) are not considered further.

From the s generators X_i, which build up the appropriate symmetry group, a general vector field

$$X = \sum_{i=1}^{s} a_i\,X_i \qquad (a_i \in \mathbb{R}) \tag{8}$$

can be constructed. For each combination of the coefficients a_i, X corresponds to an allowed transformation (in infinitesimal representation). The invariants of the transformations can be determined by integrating the characteristic equations

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}r}{\xi} = \frac{\mathrm{d}u}{\eta} \tag{9}$$

which leads to the similarity variable $z = z(r, t)$ and defines the connection between the similarity function $y(z)$ and the concentration $u(r, t)$. With knowledge of these

Table 2. Generators and one-parameter transformations.

Generator	$(u_1, r_1, t_1) = e^{\epsilon X_i}(u, r, t)$	Conditions
$X_1 = \frac{\partial}{\partial t}$	$(r, t+\epsilon, u)$	—
$X_2 = \frac{\partial}{\partial r}$	$(r+\epsilon, t, u)$	$\lambda = 1$
$X_3 = -t\frac{\partial}{\partial t} + \frac{\nu-\sigma}{2\sigma} r\frac{\partial}{\partial r} + \frac{1}{\sigma} u\frac{\partial}{\partial u}$	$\left(re^{\epsilon(\nu-\sigma)/2\sigma}, te^{-\epsilon}, ue^{\epsilon/\sigma}\right)$	$f(u) \sim u^{\sigma+1}\,(\sigma \neq 0)$
$X_4 = r\frac{\partial}{\partial r} + \frac{2}{\nu} u\frac{\partial}{\partial u}$	$\left(re^{\epsilon}, t, ue^{2\epsilon/\nu}\right)$	$m = 0$
$X_5 = -\frac{e^{dt}}{d}\frac{\partial}{\partial t} + \frac{u}{\nu}e^{dt}\frac{\partial}{\partial u}$	$\left(r, t - \frac{1}{d}\ln\left(1+\epsilon e^{dt}\right), u\left[1+\epsilon e^{dt}\right]^{1/\nu}\right)$	$d \neq 0$
$X_6 = r^2\frac{\partial}{\partial r} - 3ru\frac{\partial}{\partial u}$	$\left(\frac{r}{1-\epsilon r}, t, u(1-\epsilon r)^3\right)$	$\lambda = 1,\ \nu = -\frac{4}{3},\ m = 0$
$X_{7/8} = \frac{1}{\omega}\sin(2\tilde{r})\frac{\partial}{\partial r} - \frac{3}{2}u\cos(2\tilde{r})\frac{\partial}{\partial u}$ [a]	$\left(\frac{2}{\omega}\left[\arctan(e^{\epsilon}\tan\tilde{r}) - \frac{\varphi}{2}\right], t, ue^{-3\epsilon/2}\left[\frac{1+e^{\epsilon}\tan\tilde{r}}{1+\tan\tilde{r}}\right]^3\right)$	$\lambda = 1,\ \nu = -\frac{4}{3},\ m < 0$
$X_{7/8} = \frac{e^{\pm\omega r}}{\pm\omega}\frac{\partial}{\partial r} - \frac{3}{2}ue^{\pm\omega r}\frac{\partial}{\partial u}$ [b]	$\left(r \mp \frac{1}{\omega}\ln(1+\epsilon e^{\pm\omega r}), t, u\left[1-\epsilon e^{\pm\omega r}\right]^{3/2}\right)$	$\lambda = 1,\ \nu = -\frac{4}{3},\ m > 0$
$X_9 = r\ln r\frac{\partial}{\partial r} - 3(1+\ln r)u\frac{\partial}{\partial u}$	$\left(r^{\exp(\epsilon)}, t, ue^{-2\epsilon}r^{2(1-\exp(\epsilon))}\right)$	$\lambda = 2,\ \nu = -1,\ m = 0$

[a] $m < 0,\ \omega = \sqrt{-4m/3}$; $\tilde{r} = (\omega r + \varphi)/2$; $\varphi = \pi/2$ or 0.
[b] $m > 0,\ \omega = \sqrt{4m/3}$.

Table 3. Commutators for $f(u) = -(d/\nu)\,u + mu^{\nu+1}$, $d \neq 0$. X_2 appears only for $\lambda = 1$, X_4 only for $m = 0$.

$[X_i, X_j]$	X_1	X_2	X_4	X_5
X_1	0	0	0	dX_5
X_2	0	0	X_2	0
X_4	0	$-X_2$	0	0
X_5	$-dX_5$	0	0	0

Table 4. Commutators for $f(u) = gu^{\sigma+1}$, $\sigma \neq 0$. X_2 appears only for $\lambda = 1$.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$-X_1$
X_2	0	0	$\frac{\nu-\sigma}{2\sigma}X_2$
X_3	X_1	$\frac{\sigma-\nu}{2\sigma}X_2$	0

Table 5. Similarity variables z and similarity forms for the 10 fundamental vector fields of (3).

	$z\,(r,t)$	$u\,(r,t)$	remarks
X_1	r	$y(z)$	Stationary solution
X_2	t	$y(z)$	Homogeneous solution
$X_1 + \frac{q}{c} X_2$	$r - \frac{q}{c}\,t$	$y(z)$	Moving-wave solution
X_3	r	$t^{-1/\nu} y(z)$	Separation ($\sigma = \nu$)
	$r\,t^{\nu-\sigma/2\sigma}$	$t^{-1/\sigma} y(z)$	Scaling solution ($\sigma \neq \nu$)
$X_3 + \frac{q}{b} X_2$	$r + \frac{q}{b} \ln t$	$t^{-1/\nu} y(z)$	($\sigma = \nu$)
	$\left[r \pm \frac{2\sigma}{\nu - \sigma}\right] t^{\nu-\sigma/2\sigma}$	$t^{-1/\sigma} y(z)$	($\sigma \neq \nu$)
X_4	t	$r^{2/\nu} y(z)$	Separation
$X_1 + \frac{s}{c} X_4$	$r\,\mathrm{e}^{-(s/c)t}$	$\mathrm{e}^{(2s/\nu c)t}\, y(z)$	
X_5	r	$\mathrm{e}^{-(d/\nu)\,t}\, y(z)$	Separation
$X_2 \pm X_5$	$r \mp \mathrm{e}^{-dt}$	$\mathrm{e}^{-(d/\nu)\,t}\, y(z)$	
$X_4 \pm X_5$	$r \exp(\mp \mathrm{e}^{-dt})$	$\exp\left(\pm \frac{2}{\nu} \mathrm{e}^{-dt} - \frac{d}{\nu} t\right) y(z)$	

dependencies, the partial differential equation (3) for $u(r,t)$ can be reduced to ordinary differential equations in $y(z)$.

Especially for the first two classes of reaction terms $f(u)$ with rich symmetry, a large variety of transformations can be formed. A systematic discussion is possible by using the method of the adjoint representation outlined by Olver [14]. The adjoint representation gives an optimal system of transformations with respect to reductions. The procedure is based on the property that transformations of the symmetry group transform solutions of the differential equation into solutions. Therefore, it is sufficient to consider only reductions which lead to solutions being inequivalent with respect to symmetry transformations. By investigation of the various cases of (3), 10 fundamental vector fields built up from $X_1, \ldots, X_5$ are obtained. They are listed in table 5, together with the corresponding similarity variables z and the similarity forms connecting $y(z)$ and $u(r,t)$.

For all combinations of dimension λ and reaction terms $f(u)$, the resulting fundamental vector fields are listed in table 6 showing the optimal system of one-dimensional subalgebras for each case. The optimal system of reductions is obtained by substituting the similarity forms (cf table 5) in the partial differential equation (3). In this way, (3) is reduced to an ordinary differential equation (ODE) of the similarity function $y(z)$. Solutions $y(z)$ lead by back-substitution to so-called similarity solutions $u(r,t)$ of (3). The resulting ODEs are of second order—except for the two cases where the fundamental vector fields with $z = t$ lead to a first-order ODE. In table 7, the first-order ODEs are listed. By separating the dependent and the independent variables, they can be reduced to quadrature. For special forms of the reaction term $f(u)$ closed-form solutions are attainable. The ODEs of second order resulting from the reductions have the following general structure:

(i) for reaction terms $f(u)$ of the first two classes

Table 6. Fundamental vector fields for combinations of dimension λ and reaction terms $f(u)$.

λ	$f(u)$	Fundamental vector fields
1	$mu^{\nu+1}$	X_1, X_2, $X_1 \pm X_2$, X_3, $X_3 + \frac{q}{b}X_2$
	$-\frac{d}{\nu}u$	X_1, X_2, $X_1 + \frac{q}{c}X_2$, X_4, $X_1 + \frac{s}{c}X_4$, X_5, $X_2 \pm X_5$, $X_4 \pm X_5$
	$-\frac{d}{\nu}u + mu^{\nu+1}$	X_1, X_2, $X_1 + \frac{q}{c}X_2$, X_5, $X_2 \pm X_5$
	$gu^{\sigma+1}$	X_1, X_2, $X_1 \pm X_2$, X_3, $X_3 \pm X_2$
	Other forms	X_1, X_2, $X_1 + \frac{q}{c}X_2$
2, 3	$mu^{\nu+1}$	X_1, X_3
	$-\frac{d}{\nu}u$	X_1, X_4, $X_1 + \frac{s}{c}X_4$, X_5, $X_4 \pm X_5$.
	$-\frac{d}{\nu}u + mu^{\nu+1}$	X_1, X_5
	$gu^{\sigma+1}$	X_1, X_3
	Other forms	X_1

Table 7. Reductions to ODE of first order.

	$u(r,t)$	$f(u)$	λ	ODE
X_2	$y(t)$	Arbitrary form	1, 2, 3	$y' = f(y)$
X_4	$r^{2/\nu}\,y(t)$	$-\frac{d}{\nu}u$	1, 2, 3	$y' = \frac{2}{\nu}\left[\lambda + \frac{2}{\nu}\right] y^{\nu+1} - \frac{d}{\nu}\,y$

Table 8. ODE of second order: parameter values for $f(u) = -(d/\nu)\,u + mu^{\nu+1}$ $(\sigma = \nu \neq 0,\ \delta = m)$.

	d	m	λ	α	β	μ
X_1	Arbitrary	Arbitrary	1, 2, 3	0	$d/2$	—
$X_1 + (q/c)\,X_2$	Arbitrary	Arbitrary	1	q/c	$d/2$	0
X_3	0	$\neq 0$	1, 2, 3	0	$-1/2$	—
$X_3 + (q/b)\,X_2$	0	$\neq 0$	1	q/b	$-1/2$	0
$X_1 + (s/c)\,X_4$	$\neq 0$	0	1, 2, 3	s/c	$s/c + d/2$	1
X_5	$\neq 0$	Arbitrary	1, 2, 3	0	0	—
$X_2 \pm X_5$	$\neq 0$	Arbitrary	1	$\mp d$	0	0
$X_4 \pm X_5$	$\neq 0$	0	1, 2, 3	$\mp d$	$\mp d$	0

$$y'' + \nu\,\frac{(y')^2}{y} + \left[\alpha z^{\mu} y^{-\nu} + \frac{\lambda - 1}{z}\right] y' - \frac{2}{\nu}\,\beta\,y^{1-\nu} + \delta\,y^{\sigma+1-\nu} = 0 \tag{10}$$

where new parameters α, β and μ are introduced. The parameter δ is connected to the form of the reaction term by $\delta = m$ for $\sigma = \nu$ and $\delta = g$ for $\sigma \neq \nu$. The parameter values for all reductions of the optimal system are listed in tables 8 (with $f(u) = -(d/\nu)\,u + m\,u^{\nu+1}$) and 9 (with $f(u) = g\,u^{\sigma+1}$).

(ii) for other reaction terms $f(u)$

$$y'' + \nu\,\frac{(y')^2}{y} + \left[\frac{q}{c}y^{-\nu} + \frac{\lambda - 1}{z}\right] y' + y^{-\nu} f(y) = 0 \tag{11}$$

Table 9. ODE of second order: parameter values for $f(u) = gu^{\sigma+1}$ ($\sigma \neq 0, \nu$; $\delta = g \neq 0$).

	λ	α	β	μ
X_1	1,2,3	0	0	—
$X_1 \pm X_2$	1	± 1	0	0
X_3	1,2,3	$(\sigma - \nu)/2\sigma$	$-\nu/2\sigma$	1
$X_3 \pm X_2$	1	$(\sigma - \nu)/2\sigma$	$-\nu/2\sigma$	1

with $q\,(\lambda - 1) = 0,\ c \neq 0$. In the two equations (10) and (11) all similarity reductions from the optimal system leading to a second-order ordinary differential equation are summarized.

With respect to integrability of nonlinear evolution equations, the existence of infinitely many generalized or Lie–Bäcklund symmetries is important. Here not only transformations of the dependent and independent variables are considered but also the derivatives of the dependent variable are viewed to be independently transformable [13, 14]. This leads to more general transformations than the point transformations from the classical symmetries. Instead of (4), the corresponding generator is written in the form

$$X = \eta\,(t, r, u, u_1, u_2, \dots, u_N)\,\frac{\partial}{\partial u} \tag{12}$$

where u_j denotes the jth derivative with respect to the space coordinate and N is the order of the generalized symmetry. The existence of higher-order generalized symmetries is usually connected with the existence of a recursion operator relating symmetries of different order and therefore, leading to an infinite hierarchy of symmetries. This feature is used as criterion for a partial differential equation PDE to be integrable or exactly solvable and it seems to be connected with the possibility of transforming the nonlinear PDE to a linear one [18, 6].

For the computation of generalized symmetries from the corresponding invariance condition (5), the order N has to be fixed. Following [6] we use $N = 2n - 1$ where n is the order of the PDE, i.e. $N = 3$. Apart from the known cases with $\lambda = 1,\ \nu = -2,\ f\,(u) = 0$ [18, 6] or $\lambda = 1, \nu = -2, f\,(u) = \text{constant}$ or $\lambda = 1, \nu = -2, f\,(u) = \text{constant} \times u$ [15], no further third-order symmetries for (3) were found. This indicates that (3) with nonlinear diffusion and a nonlinear reaction term does not belong to the class of integrable or exactly solvable evolution equations.

3. Painlevé analysis, first integrals, and explicit similarity solutions

In order to solve the nonlinear second-order ODEs (10) and (11) resulting from similarity reduction, further investigations are necessary. Especially in (11), the undetermined function $f(u)$ occurs, i.e. knowledge about the reaction processes is necessary before solving it. Instead of considering special models for $f(u)$ in (11), we will focus our attention on (10) which applies to the first two classes of reaction terms, that is $f(u) = -(d/\nu)u + mu^{\nu+1}$ and $f(u) = gu^{\sigma+1}$.

Since (10) is a nonlinear differential equation, only for special values of the parameters can exact solutions be found. A useful method providing parameter values of integrable cases is the Painlevé test. This test delivers necessary conditions for the Painlevé property which is connected with the existence of the general solution in an explicit form [19,20]. An ordinary differential equation is said to possess the Painlevé property if all its solutions are meromorphic functions of the integration constants. For second-order differential equations, Painlevé and colleagues showed that every equation for $y(z)$ with the Painlevé property can be transformed by

$$y(z) = \frac{P_1(z)W(x) + P_2(z)}{P_3(z)W(x) + P_4(z)} \qquad x = x(z) \tag{13}$$

to one of 50 standard forms for $W(x)$ [16]. 44 of the 50 standard equations can be solved via a first integral by well known functions. The remaining six define new classes of transcendental functions, the so-called Painlevé transcendentals. Since (10) is a second-order differential equation, we can use the integration theory of Painlevé instead of the Painlevé test. This procedure has two main advantages: firstly, it delivers a rigorous result about the Painlevé property whereas the Painlevé test only provides necessary conditions; and secondly, it leads to the general solution in the cases with the Painlevé property.

Some sort of branch points in the solution $y(z)$ can easily be removed by a transformation

$$y(z) = \tilde{y}^{\kappa}(z) \tag{14}$$

of the dependent variable. In this way, $\tilde{y}(z)$ may possess the Painlevé property, whereas $y(z)$ does not have the Painlevé property. Hence, we will look for parameter combinations for which $\tilde{y}$ possesses the Painlevé property. A suitable choice of κ is $\kappa = -1/\nu$ leading (10) to

$$\tilde{y}'' - \left(2 + \frac{1}{\nu}\right)\frac{(\tilde{y}')^2}{\tilde{y}} + \left(\alpha z^{\mu}\tilde{y} + \frac{\lambda - 1}{z}\right)\tilde{y}' + 2\beta\tilde{y}^2 - \delta\nu\tilde{y}^{2-\sigma/\nu} = 0\,. \tag{15}$$

Especially, if $\alpha = \beta = 0$, transformation (14) with $\kappa = 1/(\nu + 1)$, $(\nu \neq -1)$ leads to

$$\tilde{y}'' + \left(\frac{\lambda - 1}{z}\right)\tilde{y}' + \delta(\nu + 1)\tilde{y}^{(\sigma+1)/(\nu+1)} = 0 \tag{16}$$

which is in case of $\sigma = \nu$ or $\sigma = -1$ a linear equation. For $\sigma \neq \nu$, (16) is an Emden equation ($\lambda = 3$) or a modified form ($\lambda = 1, 2$) [21,22].

Since (15) is of the structure

$$\tilde{y}'' = L\frac{\tilde{y}'^2}{\tilde{y}} + M(\tilde{y}, z)\tilde{y}' + N(\tilde{y}, z) \tag{17}$$

with constant L, the transformation (13) must be of the form

$$y(z) = P_1(z)W(z) + P_2(z) \qquad x = x(z)\,. \tag{18}$$

For simplicity's sake, we will assume $P_2 \equiv 0$ in (18). From an extensive discussion, it follows that the more general case with arbitary P_2 does not lead to further Painlevé cases for (15) with the admissible parameter values. Thus, the assumption $P_2 \equiv 0$ is not a restriction for the equations under consideration. By (18) with $P_2 \equiv 0$, (15) is transformed to

$$\begin{aligned} W_{xx} = {} & \left(2+\frac{1}{\nu}\right)\frac{W_x^2}{W} - \alpha\,\frac{P_1 z^{\mu}}{x'}\,W W_x + \left[-\frac{x''}{x'^2} + 2\left(1+\frac{1}{\nu}\right)\frac{P_1'}{P_1\,x'} - \frac{\lambda-1}{z\,x'}\right] W_x \\ & - \left[\alpha z^{\mu}\,\frac{P_1'}{x'^2} + 2\beta\,\frac{P_1}{x'^2}\right] W^2 + \left[-\frac{P_1''}{P_1\,x'^2} + \left(2+\frac{1}{\nu}\right)\frac{P_1'^2}{P_1^2\,x'^2} \right. \\ & \left. - \frac{\lambda-1}{z\,x'}\,\frac{P_1'}{P_1}\right] W + \delta\nu\,\frac{P_1^{1-\sigma/\nu}}{x'^2}\,W^{2-\sigma/\nu}\,. \end{aligned} \qquad (19)$$

In order to get one of the 50 standard equations, the coefficients in (19) must have special values. Hence, coupled differential equations for $P_1(z)$ and $x(z)$ follow. By suitable choices of P_1 and x and for special parameters in (15), standard forms are obtained.

The results are listed in tables 10 and 11. In those cases, (10) can be transformed to one of the 50 standard equations (in the notation of [16]). A first integral, or the general solution of the standard equations is given in [16]. Thus, for the

Table 10. Painlevé property of (15): parameter values, transformations and corresponding standard forms for $f(u) = -(d/\nu)\,u + m u^{\nu+1}$ $(\sigma = \nu \neq 0,\ \delta = m)$.

α	β	λ	ν	δ	μ	$P_1(z)$	$x(z)$	Standard form
$\neq 0$	$\neq 0$	1	$-\frac{1}{2}$	$\dfrac{-32\beta^2}{\alpha^2}$	0	$-\dfrac{4\beta}{\alpha^2}\,x(z)$	$\exp\left(-\dfrac{2\beta}{\alpha}z\right)$	V
	$\frac{1}{2}\alpha$	1, 2, 3	$-\frac{1}{2}$	0	1	$\dfrac{2}{\alpha z}$	z	V
	$\dfrac{\lambda}{4}\alpha$	1, 2, 3	$-\frac{1}{2}$	0	1	$\dfrac{4-\lambda}{\alpha}z^{-\lambda/2}$	$z^{2-\lambda/2}$	V
	$\dfrac{\lambda}{2}\alpha$	1, 2, 3	-1	0	1	1	$x' = a\,z^{1-\lambda}$	XIV
$\neq 0$	0	1	$-\frac{1}{2}$	0	0	1	$\dfrac{\alpha}{2}z$	V
		1	-1	Arbitrary	0	$\exp\left(-\dfrac{\delta}{2}z^2\right)$	z	XIV
0	$\neq 0$	1	$-\frac{1}{2}$	0	—	± 1	$\sqrt{\dfrac{\mp\beta}{3}}\,z$	II
		1	-1	0	—	1	z	XII
		2	-1	0	—	$\dfrac{1}{z^2}$	$\ln z$	XII
		1	$-\frac{2}{3}$	0	—	$-\dfrac{2}{\beta}$	z	XVIII
		1	$-\frac{2}{3}$	$\neq 0$	—	$\dfrac{2\delta}{3\beta}$	$\sqrt{-\dfrac{\delta}{3}}\,z$	XIX
		1	$-\frac{4}{5}$	0	—	$-\dfrac{3}{2\beta}$	z	XXI
		1	$-\frac{4}{5}$	$\neq 0$	—	$-\dfrac{6}{5\beta}$	$\sqrt{\dfrac{4}{5}}\,z$	XXIII
0	0	1, 2, 3	-1	Arbitrary	—	$\exp\left(-\dfrac{\delta z^2}{2\lambda}\right)$	$x' = a\,z^{1-\lambda}$	XI
		1, 2, 3	$\neq -1$	Arbitrary	—	—	—	Linear ODE

Table 11. Painlevé property of (15): parameter values, transformations and corresponding standard forms for $f(u) = gu^{\sigma+1}$ ($\sigma \neq 0, \nu;\ \delta = g \neq 0$).

α	β	λ	ν	σ	δ	μ	$P_1(z)$	$x(z)$	Standard form
$\neq 0$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{9}\alpha^2$	0	$3/\alpha$	z	VI
		1	$-\frac{1}{2}$	$\frac{1}{2}$	$-2\alpha^2$	0	$1/\alpha$	z	X
		1	$-\dfrac{n}{n+1}$	$\dfrac{n}{n+1}$	$\dfrac{n+1}{(n+2)^2}\alpha^2$	0	1	z	XXIV
		1	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{3}{4}\alpha^2$	0	1	αz	XXVIII
		1	$-\frac{3}{4}$	$\frac{3}{4}$	$-2\alpha^2$	0	1	$\frac{3}{2}\alpha z$	XXXV
0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\neq 0$	—	1	$\dfrac{\sqrt{-\delta}}{2} z$	VII
		1	-1	$-3, -2, 1$	$\neq 0$	—	1	z	XII
		2	-1	$-3, -2, 1$	$\neq 0$	—	$z^{-2/(\sigma+1)}$	$\ln z$	XII
		1	$-\frac{4}{5}$	$-\frac{8}{5}$	$\neq 0$	—	$\frac{4}{5}\delta$	z	XXII
		1	$-\frac{2}{3}$	$\frac{2}{3}$			1	$\frac{2}{3}\sqrt{-\delta}\, z$	XXIX
		1	$-\frac{2}{3}$	-2			1	$2\sqrt{\dfrac{\delta}{3}}\, z$	XXXII
		1, 2, 3	$\neq -1$	$\neq \nu$	Arbitrary	—	—	—	Mod. Emden

parameter values listed in tables 10 and 11, the explicit general solutions are known. A large variety of integrable cases are due to $\alpha = \beta = 0$, i.e. reductions where the similarity variable z is the space variable r. If the reaction term is given by $f(u) = -(d/\nu)u + mu^{\nu+1}$, the similarity reductions can be transformed to a linear equation ($\nu \neq -1$) or to the standard form XI ($\nu = -1$). In the other case in which $f(u) = gu^{\sigma}$, an Emden equation or a modified Emden equation is obtained for $\nu \neq -1$. Apart from the cases with $\alpha = \beta = 0$, the transformation to standard form is only possible for a few, very special parameter combinations. Restrictions have to be applied on both the parameters occuring in the differential equation, and the group parameter of the similarity reduction. Further, all of those Painlevé cases correspond to processes with 'fast diffusion' where $\nu < 0$.

A definition of the Painlevé property for PDE and a corresponding Painlevé test was proposed by Weiss *et al* [17]. A PDE is said to have the Painlevé property if its general solution is single-valued about an arbitrary singular manifold. Performing this type of Painlevé test to the reaction diffusion equation (3) with $f(u) = -(d/\nu)u + mu^{\nu+1}$ or $f(u) = gu^{\sigma+1}$, we can take the invariance of the Painlevé property under similarity reductions [8] into account. Therefore, only such parameters leading to Painlevé type ODE by similarity reduction have to be considered. It turns out that the Painlevé test of the PDE fails for all values of $\lambda, \nu, d, m, \sigma$, and g. Hence, although the PDE does not possess the Painlevé property and is thus supposed to be not integrable, similarity reductions with the Painlevé property exist. We note that the cases admitting an infinite number of Lie–Bäcklund symmetries do not possess the Painlevé property but some sort of weak Painlevé property [20] with half-integer resonances indicating that the original Painlevé property is a too restrictive criterion for integrability even in the area of PDE.

Apart from the Painlevé cases, there are further combinations of the parameters in (10) for which first integrals exist. In order to find first integrals, we multiply (10) by $h(z)y^{2\nu}y' + g(z)y^{\gamma}$ with arbitrary functions g, h and arbitrary γ. Collecting the terms which form a total differential, we find an integral if the other terms cancel

Table 12. First integrals of (10): parameters for which a first integral C exists ($\alpha \neq 0$).

ν	σ	μ	λ	β	δ	$g(z)$	C
Arbitrary	-1	0	1	0	Arbitrary	1	(21)
		1	$1,2,3$	$-\nu\alpha$	Arbitrary	z	or
		1	$1,2,3$	$-\frac{\lambda\nu\alpha}{2}$	Arbitrary	$z^{\lambda-1}$	(22)
$\neq -1$	ν	0	1	Arbitrary	$-\frac{4\beta^2}{\alpha^2\nu^2(\nu+1)}$	$\exp\left(-\frac{2\beta z}{\alpha\nu}\right)$	(20)
		1	$1,2,3$	$-\nu\alpha$	0	z	
		1	$1,2,3$	$-\frac{\lambda\nu\alpha}{2}$	0	$z^{\lambda-1}$	

out. To reduce the cases which have to be discussed, we do not consider cases with $\alpha = \beta = 0$, $\sigma = -1$ and $\alpha = \beta = 0$, $\sigma = \nu$. They can either be tranformed by $y = \tilde{y}^{1/(\nu+1)}$ to a linear equation ($\nu \neq -1$) or be reduced to standard equation XI of [16] ($\nu = -1$) which is completely integrable, too.

In the case of $h(z) \equiv 0$ integrals of the form

$$C = g(z)y^\nu y' + \left((\lambda-1)\frac{g(z)}{z} - g'(z)\right)\frac{y^{\nu+1}}{\nu+1} + \alpha z^\mu g(z)y \tag{20}$$

for $\sigma = \nu \neq -1$,

$$C = g(z)y^\nu y' + \left((\lambda-1)\frac{g(z)}{z} - g'(z)\right)\frac{y^{\nu+1}}{\nu+1} + \alpha z^\mu g(z)y + \delta\int^z g(\tilde{z})\,\mathrm{d}\tilde{z} \tag{21}$$

for $\sigma = -1, \nu \neq -1$, and

$$C = g(z)\frac{y'}{y} + \left((\lambda-1)\frac{g(z)}{z} - g'(z)\right)\log(y) + \alpha z^\mu g(z)y + \delta\int^z g(\tilde{z})\,\mathrm{d}\tilde{z} \tag{22}$$

for $\sigma = \nu = -1$ are found where the admissable parameters and the function $g(z)$ are given in table 12. If $g(z) \equiv 0$, the integral

$$C = \frac{1}{2}y^{2\nu}y'^2 - \frac{2\beta}{\nu}\frac{y^{\nu+2}}{\nu+2} + \delta\frac{y^{\sigma+\nu+2}}{\sigma+\nu+2} \tag{23}$$

is obtained for $\alpha = 0$ and $\lambda = 1$. In (23) a logarithmic term occurs instead of the power term for $\nu = -2$ or $\nu = -2-\sigma$, i.e. when the denominator vanishes. With $g(z) \not\equiv 0$ and $h(z) \not\equiv 0$ two further combinations of parameters possessing a first integral can be detected. Both are related to $\nu = -1$ and $\lambda = 2$. For $\alpha = 0,\ \sigma = \nu = -1,\ \lambda = 2,\ \delta = 0$, the integral reads

$$C = \tfrac{1}{2}z^2y^{-2}y'^2 + 2zy^{-1}y' + 2\beta z^2 y \tag{24}$$

and in case of $\alpha = \beta = 0,\ \nu = -1,\ \sigma \neq -1,\ \lambda = 2$,

$$C = \frac{1}{2}z^2y^{-2}y'^2 + \frac{2}{\sigma+1}zy^{-1}y' + \delta z^2\frac{y^{\sigma+1}}{\sigma+1} \tag{25}$$

is obtained.

By comparison with the Painlevé cases (tables 10 and 11), we note that for all cases with Painlevé property a first integral can be found by using this direct method. The only exceptions are the cases with $\alpha \neq 0,\ \beta = 0$. There, the integrals have a different structure which is not within reach of the ansatz. However, in this way we get first integrals of a wider class of diffusion reaction equations than by Painlevé analysis. For example, in case of $\alpha \neq 0,\ \beta \neq 0$, integrals are attained for arbitrary values of ν whereas the Painlevé property is given only for $\nu = -\frac{1}{2}$ or $\nu = -1$. With regard to the first integration of the second-order ordinary differential equation (10), the construction of first integrals covers a larger range of parameter values than the Painlevé analysis.

On the other hand, the Painlevé cases allow an explicit second integration, which is in general not possible for the first integrals constructed here. Only in some cases can the integration be reduced to a quadrature or performed explicitly for arbitrary constants C. In many other cases, the second integration can be done for special values of the constant C, e.g. for $C = 0$.

Next, we will focus our attention on explicit solutions. Because of the large number of different cases, it is not advisable to treat every case. Hence, we will consider two illustrative examples, for one of which the general solution can be given as a consequence of the Painlevé property, the other possessing a first integral which allows the second integration for the special value $C = 0$ of the integration constant. Although both examples are related to the reduction to the moving-wave variable $z = r - vt$, exact solutions can also be obtained from the above results (tables 5–12) for more complicated similarity variables. Comparison with table 8 shows that the moving-wave reduction exists only in one-dimensional problems ($\lambda = 1$) and that for non-vanishing propagation velocity $v = q/c$, the parameters $\alpha \neq 0$ and $\mu = 0$ have to be used.

In the Painlevé cases, a transformation to the standard forms V and XIV is possible (table 10). Especially for $\alpha = v \neq 0,\ \beta = d/2 = 0,\ \lambda = 1,\ \nu = -1$, equation (10) can be tranformed by $y(z) = \exp(-\delta x^2/2)\, W(x),\ x = z$ to form XIV leading to

$$u(r,t) = \frac{1}{v}\sqrt{\frac{2m}{\pi}}\frac{\exp\left(-\tilde{z}^2\right)}{\mathrm{erf}(\tilde{z}) + K}\,. \tag{26}$$

In (26), erf is the error function, K the second integration constant, and $\tilde{z} = \sqrt{m/2}(r-vt) - C/\sqrt{2m}$. For $K > 1$, solution (26) has the shape of a pulse. Hence, (26) is a pulse-like moving-wave solution of the nonlinear diffusion equation (3) with $\nu = -1$ and $f(u) = m =$ constant. The typical behaviour of this solution is plotted in figure 1.

In the second example, we consider the equation in one space dimension

$$u_t = (u^\nu u_x)_x + u(1 - u^\nu) \tag{27}$$

which is a generalization of the famous Fisher equation describing, e.g., the propagation of an advantageous gene in population genetics. By consulting table 12, one recognizes that

$$C = \exp\left(\frac{z}{v}\right)\left(y^\nu y' - \frac{1}{v}\frac{y^{\nu+1}}{\nu+1} + vy\right) \tag{28}$$

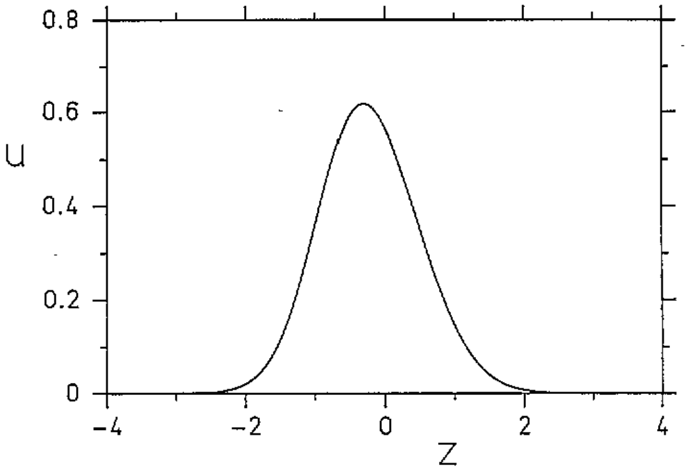

Figure 1. Moving-wave solution (26) with $C = 0,\ v = 1,\ m = 2,\ K = 2$.

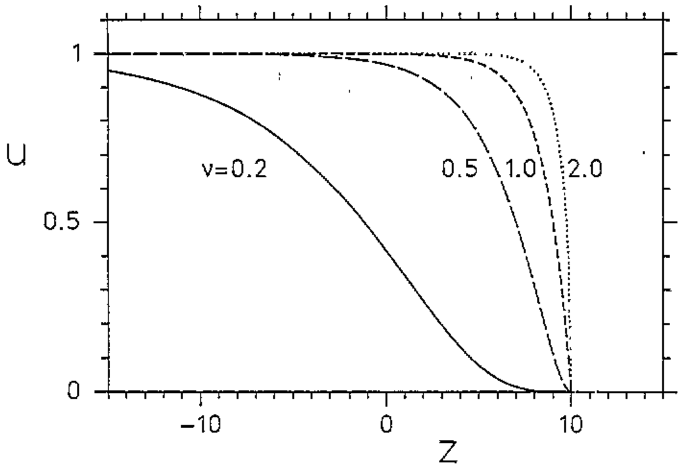

Figure 2. Moving-wave solution (29) with $z_0 = 10$ for various values of ν.

is a first integral for $\alpha^2 = v^2 = 1/(\nu + 1)$, $(\nu > -1)$. The second integration cannot be performed for arbitrary C. However, for $C = 0$, the transformation $y = \tilde{y}^{-1/\nu}$ leads to a Riccati equation. Since $u(x,t) \equiv 0$ solves (27), one finds a kink-shaped solution

$$u(x,t) = \begin{cases} (1 - \exp[\nu v(x - vt - z_0)])^{1/\nu} & x - vt \leqslant z_0 \\ 0 & x - vt \geqslant z_0 \end{cases} \tag{29}$$

with $v^2 = 1/(\nu + 1)$, $(\nu > 0)$ and the integration constant z_0. For $\nu > 1$, however, (29) is a continuous but, in $x - vt = z_0$, non-differentiable solution. The moving-wave solution (29) of the generalized Fisher equation previously found by Newman [23] is plotted in figure 2.

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